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Stochastic Quantization Approach for the Ising Model

A. Bérard[1] and Y. Grandati[1]

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The Ising model is studied in the fermionic formulation of the stochastic quantization. An exact stochastic equation is given for $D = 2$ and 3 and in a Hartree approximation a method is developed for treating the two-point correlation functions.

1. INTRODUCTION

The stochastic quantization method of Parisi and Wu[(1)] provides a novel and interesting connection between Euclidean quantum field theory and classical statistical mechanics. The application of this formalism to theories containing fermionic fields has been treated using Grassmann variables developed by Berezin.[(2)]

The fact is that the Ising model, one of the few nontrivial integrable models of statistical physics, can also be interpreted in terms of Grassmann variables.[(3)] The most important aspect of this approach is that all Grassmann correlation functions may be easily calculated in two dimensions, a result which is impracticable with spin variables. We have at our disposition two types of variables: spin variables, which have a simple physical interpretation, but are very difficult to work with mathematically, and Grassmann variables, which do not have as a simple physical interpretation, but are easily to use.

Here we apply the stochastic quantization method to this model. In Section 2 we briefly review the fermionic formalism of the stochastic quantization and the two-dimensional Ising model explained in Grassmann variables. Then we treat and calculate all the correlation functions of this model

[1] L.P.L.I. Institut de Physique, F-57070, Metz, France.

in Section 3 for two dimensions, and in Section 4 for three dimensions in a kind of Hartree approximation.

2. BASIC FORMALISM

2.1. Stochastic Quantization

The basic idea of stochastic quantization is to consider the Euclidean path integral measure as a stationary distribution of a stochastic process in an extra variable, the fictitious time t. The diffusion toward equilibrium in $(d+1)$ dimensions is described for the fermionic theories through two Langevin equations:

$$\frac{\partial\psi_\alpha(x,t)}{\partial t} = -\frac{\delta S_E[\psi,\bar{\psi}]}{\delta\bar{\psi}_\alpha[x,t]} + \theta_\alpha(x,t)$$

$$\frac{\partial\bar{\psi}_\alpha(x,t)}{\partial t} = \frac{\delta S_E[\psi,\bar{\psi}]}{\delta\psi_\alpha[x,t]} + \bar{\theta}_\alpha(x,t) \tag{1}$$

where $S_E[\psi, \bar{\psi}]$ stands for the action, which depends on the fermionic fields ψ and $\bar{\psi}$.

Working in Euclidean space, fields have to be treated as independent Grassmann variables. Then stochastic expectation values of the Gaussian noise are defined by

$$\langle\theta\rangle_\theta = \langle\bar{\theta}\rangle = 0$$

$$\langle\theta_\alpha(x,t)\bar{\theta}_\beta(x',t')\rangle_\theta = 2\delta_{\alpha\beta}\delta^D(x-x')\delta(t-t') \tag{2}$$

and similarly for higher n-point functions, taking into account the anticommutating nature of the noise fields $\theta(x,t)$ and $\bar{\theta}(x,t)$.

The first important characteristic of the stochastic quantization is that we can start from the "exact" equation of motion itself, while the conventional quantization procedures, canonical quantization and Feynmann path integral methods, cannot be applied to any dynamical system which has no Hamiltonian or Lagrangian.

2.2. Two-Dimensional Ising Model

It is well known[6] that for a large number of statistical mechanical systems which have a graphical interpretation, the partition function has a representation in terms of anticommutative variables. The two-dimensional Ising model has such an interpretation where the sum is over closed nonoverlapping but intersecting polygonal curves. This is obtained by drawing curves separating the region of up spin from down spin.

After calculation we obtain in the momentum space[4,5]

$$Z = \cosh(\beta J)^{2N} \int D\eta_p \, D\eta^t_{-p} \exp(S_E[\eta, \eta^t]) \tag{3}$$

which is expressed by means of 4-component Grassmann variables η_p and η_{-p} with the definition

$$\eta_p = \begin{pmatrix} (\eta^V)^t_{-p} \\ (\eta^V)_p \\ (\eta^H)^t_{-p} \\ (\eta^H)_p \end{pmatrix}$$

where the four components are introduced at each site of the reciprocal lattice, and can be graphically represented by

$$\underset{(\eta^H)^t_{-p}}{{}_p\!\mapsto} \quad \underset{(\eta^H)_p}{\mapsto_p} \quad \underset{(\eta^V)^t_{-p}}{\uparrow_p} \quad \underset{(\eta^V)_p}{\uparrow^p}$$

The functional action is given by

$$S_E[\eta, \eta^t] = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \, \eta^t_{-p} D_{-p,p} \eta_p \tag{4}$$

where

$$D_{-p,p} = \begin{pmatrix} 0 & \alpha & 1 & 1 \\ -\alpha^* & 0 & -1 & 1 \\ -1 & 1 & 0 & \beta \\ -1 & -1 & -\beta^* & 0 \end{pmatrix} \tag{5}$$

with

$$\alpha = 1 + K\exp(ipx)$$
$$\beta = 1 + K\exp(-ipx)$$
$$K = \tanh(\beta J)$$

A straightforward calculation permits us to reobtain the famous Onsager result for the free energy.

3. TWO-DIMENSIONAL ISING MODEL IN STOCHASTIC QUANTIZATION

From the preceding action we can write the following Langevin equations:

$$\frac{\partial \eta_p^i(t)}{\partial t} = -\frac{1}{2} D_{-p,p}^{ij} \eta_p^j(t) + \theta_p^i(t)$$
$$\frac{\partial \eta_{-p}^{ti}(t)}{\partial t} = -\frac{1}{2} \eta_{-p}^{tj}(t) D_{p,-p}^{ij} + \theta_{-p}^{ti}(t) \tag{6}$$

with

$$\langle \theta_p^i(t) \theta_{-p'}^{tj}(t') \rangle_\theta = 2\delta^{ij} \delta^2(p - p') \delta(t - t') \tag{7}$$

The D matrix can be diagonalized as

$$D_{-p,p}^d = \begin{pmatrix} i\lambda & 0 & 0 & 0 \\ 0 & -i\lambda & 0 & 0 \\ 0 & 0 & i\lambda' & 0 \\ 0 & 0 & 0 & -i\lambda' \end{pmatrix}$$

where

$$\lambda = \sqrt{2 + (1 + K^2) + K(\cos(p_x) + \cos(p_y)) - \sqrt{4 + K(\cos(p_x) + \cos(p_y))^2 + 4K^2 - 8}}$$
$$\lambda' = \sqrt{2 + (1 + K^2) + K(\cos(p_x) + \cos(p_y)) + \sqrt{4 + K(\cos(p_x) + \cos(p_y))^2 + 4K^2 - 8}}$$

But in the $t \to \infty$ limit, we can easily see that we do not recover the usual Green functions. This problem is well known in stochastic quantization.[7] Basically this evidences the fact that there exists no classical analogue of fermion fields. The consequence here is the appearance of operators which are not positive definite. The idea to solve this problem is to choose a bosonized version of the two Langevin equations by the introduction of kernels.[7]

In view to obtaining more simplified Langevin equations, we take the following kernels:

$$K_p = (D^d)_{-p,p}^{-1}$$

Then we have

$$\frac{\partial \eta_p^i(t)}{\partial t} = -\frac{1}{2} \eta_p^i(t) + \theta_p^i(t)$$
$$\frac{\partial \eta_{-p}^{ti}(t)}{\partial t} = -\frac{1}{2} \eta_{-p}^{ti}(t) + \theta_{-p}^{ti}(t) \tag{8}$$

and after a simple matrix manipulation

$$\langle \theta^i_p \theta^{tj}_{-p} \rangle = \lim_{t \to \infty} \langle \theta^i_p(t) \theta^{tj}_{-p}(t) \rangle_\theta = (D^d)^{-1}_{-p,p} \tag{9}$$

In position space we then obtain

$$\langle \eta^i_r \eta^{tj}_{r'} \rangle = \int \frac{d^2p}{(2\pi)^2} (D^{-1})^{ij}_{-p,p} \exp(ip(r - r')) \tag{10}$$

where

$$(D^{-1})_{-p,p} =$$

$$\begin{pmatrix} \beta - \beta^* & \beta + \beta^* - \alpha\beta\beta^* & -2 + \alpha\beta^* & -2 + \alpha\beta \\ -\beta - \beta^* + \alpha\beta\beta^* & -\beta + \beta^* & 2 - \alpha^*\beta^* & -2 - \alpha^*\beta \\ 2 - \alpha^*\beta & -2 + \alpha\beta & -\alpha + \alpha^* & -\alpha - \alpha^* - \alpha\alpha^*\beta \\ 2 - \alpha^*\beta^* & 2 - \alpha\beta^* & -\alpha - \alpha^* + \alpha\alpha^*\beta^* & \alpha - \alpha^* \end{pmatrix}$$

We can give the 12 possible correlation functions founded by Samuel[4] in another context, for instance, we have

$$\begin{aligned} \langle (\eta^V)_r (\eta^V)^t_{r'} \rangle &= \int \frac{d^2p}{(2\pi)^2} \exp(ip(r - r')) \left(\frac{\alpha + \alpha^* - \alpha\alpha^*\beta^*}{\det D} \right) \\ \langle (\eta^H)_r (\eta^H)^t_{r'} \rangle &= \int \frac{d^2p}{(2\pi)^2} \exp(ip(r - r')) \left(\frac{\beta + \beta^* - \alpha^*\beta\beta^*}{\det D} \right) \\ \langle (\eta^V)_r (\eta^H)^t_{r'} \rangle &= \int \frac{d^2p}{(2\pi)^2} \exp(ip(r - r')) \left(\frac{-2 - \alpha^*\beta^*}{\det D} \right) \\ \langle (\eta^H)_r (\eta^V)^t_{r'} \rangle &= \int \frac{d^2p}{(2\pi)^2} \exp(ip(r - r')) \left(\frac{-2 - \alpha^*\beta^*}{\det D} \right) \end{aligned} \tag{11}$$

with

$$\det D = (1 + K^2)^2 - 2K(1 - K^2)(\cos(p_x) + \cos(p_y))$$

4. A KIND OF HARTREE APPROXIMATION FOR THE THREE-DIMENSIONAL ISING MODEL IN STOCHASTIC QUANTIZATION

Here we concentrate on the three-dimensional Ising model. Its dual is the Z_2 three-dimensional gauge model with similar variables assigned to links and interacting on elementary plaquettes. The fermionization of this theory was also realized by Samuel[4] and Itzykson and Drouffe,[5] who generalized the two-dimensional formalism in the following partition function:

$$Z = \cosh(\beta J)^{3N} \int \prod_x d\eta_x^1 d\eta_x^{+1} d\eta_x^2 d\eta_x^{+2} d\eta_x^3 d\eta_x^{+3} \exp(S) \tag{12}$$

where

$$S = S_{\text{plaquettes}} + S_{\text{corners}} + S_{\text{vertex}} \tag{13}$$

with

$$S_{\text{plaquettes}} = K \sum_x \sum_{\substack{i<j \\ i \neq j \neq k}} \omega_{ijk} \eta^{+j}_{x+v_{ij}} \eta^{j}_{x+u_{ij}} \eta^{+k}_{x+v_{ik}} \eta^{k}_{x+u_{ik}} \tag{14}$$

$$S_{\text{corners}} = \sum_x \sum_{i \neq j} \{\eta^{+i}_{x+u_{ij}} \eta^{j}_{x+u_{ij}} + \eta^{i}_{x+u_{ik}} \eta^{+j}_{x+u_{ik}} + \eta^{+i}_{x+u_{ij}} \eta^{+j}_{x+u_{ij}} + \eta^{i}_{x+u_{ij}} \eta^{i}_{x+u_{ij}}\} \tag{15}$$

$$S_{\text{vertex}} = \sum_x \sum_{i \neq j} \eta^{i}_{x+u_{ij}} \eta^{+i}_{x+u_{ij}} \tag{16}$$

with

$$\omega_{ijk} = 1 \qquad \text{for} \quad i \neq j \neq k$$

$$u_{ij} = \frac{1}{2}(e_i + e_j)$$

$$v_{ij} = \frac{1}{2}(e_i - e_j)$$

where e_i is the unitary vector in the i direction.

Since the plaquette terms are quartic, the partition function is no longer integrable and we need to develop approximation schemes. First we attend to the quadratic terms, whose contribution to the partition function is Gaussian.

4.1. The Quadratic Terms

These terms can be easily rewritten in the form

$$S_{\text{corners}} + S_{\text{vertex}} = \int \frac{d^3p}{(2\pi)^3} \sum_{i,j=1}^{6} \eta^{ti}_{-p} M^{ij}_{-p,p} \eta^{j}_{p} \tag{17}$$

where η^{ti}_{-p} is the transpose of η^{i}_{p}, given by

$$\eta_p^i = \begin{pmatrix} \eta_{-p}^{+1} \\ \eta_p^1 \\ \eta_{-p}^{+2} \\ \eta_p^2 \\ \eta_{-p}^{+3} \\ \eta_p^3 \end{pmatrix}$$

The M matrix is given by

$$M^{ij}_{-p,p} = \frac{1}{2} \begin{pmatrix} 0 & -2 & -a_3^* & -1 & -a_2^* & -1 \\ 2 & 0 & 1 & -a_3 & 1 & -a_2 \\ a_3^* & -1 & 0 & -2 & -a_1^* & -1 \\ 1 & a_3 & 2 & 0 & 1 & -a_1 \\ a^*{}_2 & -1 & a_1^* & -1 & 0 & -2 \\ 1 & a_2 & 1 & a_1 & 2 & 0 \end{pmatrix}$$

with $a_j = \varepsilon_{jkl} \exp(2ipu_{kl})$, where there is no summation on the indices.

In first approximation let us suppress the quartic terms (that is, let us take $K = 0$). The Langevin equations associated with the preceding action is then

$$\frac{\partial \eta_p^i(t)}{\partial t} = -M^{ij}_{-p,p}\eta_p^j(t) + \theta_p^i(t)$$

$$\frac{\partial \eta_p^{ti}(t)}{\partial t} = -\eta_p^{tj}(t)M^{ji}_{p,-p} + \theta_p^i(t) \tag{18}$$

The M matrix is antisymmetric and its eigenvalues are pure imaginary complex conjugate numbers. Then in the $t \to \infty$ limit, we recover the same problem as in two dimensions. As before, this difficulty is avoided by the introduction of a kernel:

$$\begin{cases} \dfrac{\partial \eta_p^i(t)}{\partial t} = -\eta_p^i(t) + \theta_p^i(t) \\ \dfrac{\partial \eta_p^{ti}(t)}{\partial t} = -\eta_p^{ti}(t) + \theta_p^i(t) \end{cases}$$

with now

$$\langle \theta_p^i(t)\theta_{-p}^{tj}(t')\rangle_\theta = 2(M^{-1})_{-p,p}^{ij}\delta(t - t')$$

where

$$M^{-1} = \frac{1}{2D}\begin{pmatrix} 0 & -2 & E_3 & (1 - a_1^* a_2) & -E_2 & (1 + a_1^* a_3) \\ 2 & 0 & -(1 - a_i a_2^*) & E_3^* & -(1 + a_1 a_3^*) & -E_2^* \\ -E_3 & 1 - a_1 a_2^* & 0 & -2 & E_1 & 1 - a_2^* a_3 \\ -(1 - a_1^* a_2) & -E_3^* & 2 & 0 & -(1 - a_2 a_3^*) & E_1^* \\ E_2 & 1 + a_1 a_3^* & -E_1 & 1 - a_2 a_3^* & 0 & -2 \\ -(1 + a_1^* a_3) & E_2^* & -(1 - a_2^* a_3) & -E_1^* & 2 & 0 \end{pmatrix}$$

and

$$E_1 = 2a_1 - a_2 + a_3$$
$$E_2 = a_1 - 2a_2 + a_3$$
$$E_3 = a_1 - a_2 + 2a_3$$
$$D = a_1^* E_1 - a_2^* E_2 + a_3^* E_3 - 4$$

4.2. The Quartic Term

In momentum space the action has a more complicated form. We can write it as

$$S_{\text{plaquettes}} = K \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3} \times \{\psi_{-p1}^{+1}(t)\psi_{p2}^{1}(t)\psi_{-p3}^{+2}(t)\psi_{p1-p2+p3}^{2}(t)\exp[i(U_{12}p_1 + V_{12}p_2 + e_2p_3)] \tag{20}$$

$$+ \psi_{-p1}^{+2}(t)\psi_{p2}^{2}(t)\psi_{-p3}^{+3}(t)\psi_{p1-p2+p3}^{3}(t)\exp[i(U_{23}p_1 + V_{23}p_2 + e_3p_3)] \tag{21}$$

$$+ \psi_{-p1}^{+3}(t)\psi_{p2}^{3}(t)\psi_{-p3}^{+1}(t)\psi_{p1-p2+p3}^{1}(t)\exp[i(U_{31}p_1 + V_{31}p_2 + e_1p_3)]\} \tag{22}$$

Introducing the vector η_p^i of the last section, we obtain

$$S_{\text{plaquettes}} = K \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3} \times \{\eta_{-p1}^{t1}(t)\eta_{p2}^{2}(t)\eta_{-p3}^{t3}(t)\eta_{p1-p2+p3}^{4}(t) \tag{23}$$

$$\exp[i(U_{12}p_1 + V_{12}p_2 + e_2p_3)]$$
$$+ \eta^{t3}_{-p_1}(t)\eta^4_{p_2}(t)\eta^{t5}_{-p_3}(t)\eta^6_{p_1-p_2+p_3}(t)$$
$$\exp[i(U_{12}p_1 + V_{12}p_2 + e_2p_3)] \tag{24}$$
$$+ \eta^{t5}_{-p_1}(t)\eta^6_{p_2}(t)\eta^{t1}_{-p_3}(t)\eta^2_{p_1-p_2+p_3}(t)$$
$$\exp[i(U_{12}p_1 + V_{12}p_2 + e_2p_3)]\} \tag{25}$$

This expression can be written in a more compact form as

$$S_{\text{plaquettes}} = K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}$$
$$\times \sum_{k,l\in(12,23,31)} \eta^{t2k+1}_{-p_1}(t)\eta^{2k}_{p_2}(t)\eta^{t2l-1}_{-p_3}(t)\eta^{2l}_{p_1-p_2+p_3}(t)$$
$$\exp[i(U'_{kl}p_1 + V'_{kl}p_2 + e'_l p_3)] \tag{26}$$

where

$$e'_1 = e_1; \qquad e'_3 = e_2; \qquad e'_5 = e_3$$
$$U'_{kl} = \frac{e'_k + e'_l}{2}; \qquad V'_{kl} = \frac{e'_k - e'_l}{2}$$

Finally we obtain for the Langevin equations

$$\frac{\partial\eta^1_p(t)}{\partial t} = -M^{1j}_{-p,p}\eta^j_p(t) - K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\{\eta^2_{p_1}(t)\eta^{t3}_{-p_2}(t)\eta^4_{p-p_1+p_2}(t)$$
$$\exp[i(U_{12}p + V_{12}p_1 + e_2p_2)]$$
$$+ \eta^{t5}_{-p_1}(t)\eta^6_{p_2}(t)\eta^2_{p_1-p_2+p}(t)$$
$$\exp[i(U_{31}p_1 + V_{31}p_2 + e_1p)] + \theta^1_p(t)$$

$$\frac{\partial\eta^2_p(t)}{\partial t} = -M^{2j}_{-p,p}\eta^j_p(t) + \theta^2_p(t)$$

$$\frac{\partial\eta^3_p(t)}{\partial t} = -M^{3j}_{-p,p}\eta^j_p(t) - K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\{\eta^4_{p_1}(t)\eta^{t5}_{-p_2}(t)\eta^6_{p-p_1+p_2}(t)$$
$$\exp[i(U_{23}p + V_{23}p_1 + e_3p_2)]$$
$$+ \eta^{t1}_{-p_1}(t)\eta^2_{p_2}(t)\eta^4_{p_1-p_2+p}(t)$$
$$\exp[i(U_{12}p_1 + V_{12}p_2 + e_2p)]\} + \theta^3_p(t)$$

$$\frac{\partial \eta_p^4(t)}{\partial t} = -M^{4j}_{-p,p}\eta_p^j(t) + \theta_p^4(t) \tag{27}$$

$$\frac{\partial \eta_p^5(t)}{\partial t} = -M^{5j}_{-p,p}\eta_p^j(t) - K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\{\eta^{t3}_{-p_1}(t)\eta^4_{p_2}(t)\eta^6_{p_1-p_2+p}(t)$$
$$\exp[i(U_{23}p_1 + V_{23}p_2 + e_3p)]$$
$$+ \eta^6_{p_1}(t)\eta^{t1}_{-p_2}(t)\eta^2_{p-p_1+p_2}(t)\exp[i(U_{31}p + V_{31}p_1 + e_1p_2)]\} + \theta_p^5(t)$$

$$\frac{\partial \eta_p^6(t)}{\partial t} = -M^{6j}_{p,-p}\eta_p^j(t) + \theta_p^6(t)$$

Similarly we have for the transposed fields

$$\frac{\partial \eta^{t1}_{-p}(t)}{\partial t} = -\eta^{tj}_{-p}(t)M^{j1}_{p,-p} + \theta^{t1}_{-p}(t)$$

$$\frac{\partial \eta^{t2}_{-p}(t)}{\partial t} = -\eta^{j}_{p}(t)M^{j2}_{p,-p} - K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\{\eta^{t1}_{-p_1}(t)\eta^{t3}_{-p_2}(t)\eta^4_{p_1-p+p_2}(t)$$
$$\exp[i(U_{12}p_1 + V_{12}p + e\ldots)]$$
$$+ \eta^{t5}_{-p_1}(t)\eta^6_{p_2}(t)\eta^{t1}_{p_1-p_2-p}(t)\exp[i(U_{31}p_2 + e_3p_1 + e_1p)]\} + \theta^{t2}_{-p}(t)$$

$$\frac{\partial \eta^{t3}_{-p}(t)}{\partial t} = -\eta^{tj}_{-p}(t)M^{j3}_{p,-p} + \theta^{t3}_{-p}(t) \tag{28}$$

$$\frac{\partial \eta^{t4}_{-p}(t)}{\partial t} = -\eta^{j}_{p}(t)M^{j4}_{p,-p} - K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\{\eta^{t1}_{-p_1}(t)\eta^2_{p_2}(t)\eta^{t3}_{p_1-p_2-p}(t)$$
$$\exp[i(U_{12}p_2 + e_1p_1 + e\ldots)]$$
$$+ \eta^{t3}_{-p_1}(t)\eta^{t5}_{-p_2}(t)\eta^6_{p_1-p+p_2}(t)\exp[i(U_{23}p_1 + V_{23}p + e_3p_2)]\} + \theta^{t4}_{-p}(t)$$

$$\frac{\partial \eta^{t5}_{-p}(t)}{\partial t} = -\eta^{tj}_{-p}(t)M^{j5}_{p,-p} + \theta^{t5}_{-p}(t)$$

$$\frac{\partial \eta^{t6}_{-p}(t)}{\partial t} = -\eta^{j}_{p}(t)M^{j6}_{p,-p} - K\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\{\eta^{t3}_{-p_1}(t)\eta^4_{p_2}(t)\eta^{t5}_{-p+p_1-p_2}(t)$$
$$\exp[i(U_{23}p_2 + e_2p_1 + \ldots)]$$
$$+ \eta^{t5}_{-p_1}(t)\eta^{t1}_{-p_2}(t)\eta^2_{p-p_1+p_2}(t)\exp[i(U_{31}p_1 + V_{31}p + e_1p_2)]\} + \theta^{t6}_{-p}(t)$$

This can be rewritten in the form

$$\frac{\partial \eta_p^i(t)}{\partial t} = -M^{ij}_{-p,p}\eta_p^j(t) - 2K \sum_{l=1}^{6} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \delta'_{il}$$
$$\times\ \eta_{p_1}^{i+1}(t)\eta^{tl}_{-p_2}(t)\eta^{l+1}_{p-p_1+p_2}(t)$$
$$\exp[i(U_{i'l'}p + \varepsilon_{i'l'}V'_{l'l'}p_1 + e'_{l'}p_2)] + \theta_p^i(t) \tag{29}$$

where ε_{ij} is the usual antisymmetric tensor, the index i' is defined as $i' = (i + 1)/2$, and

$$\delta'_{il} = (1 - \delta_{l2})(1 - \delta_{l4})(1 - \delta_{l6})(1 - \delta_{li})(1 - \delta_{i2})(1 - \delta_{i4})(1 - \delta_{i6})$$

In order to illustrate the possibilities of the approach developed here, we can solve the preceding equations in a kind of Hartree-type approximation. This can be made by replacing

$$\eta_{p_1}^i(t)\eta^{tl}_{-P_2}(t')\eta^k_{p_3}(t'') \mapsto \langle \eta_{p_1}^i(t)\eta^{tl}_{-p_2}(t')\rangle_\theta \eta^k_{p_3}(t'') - \langle \eta^k_{p_3}(t''\eta^t)^l_{-p_2}(t')\rangle_\theta \eta^i_{p_1}(t)$$

where the mean values are determined at the zero order of perturbation. Then the Langevin equations are written

$$\frac{\partial \eta_p^1(t)}{\partial t} = -M^{1j}_{-p,p}\eta_p^j(t) - K[1/2 \exp(iU_{12}p)\eta_p^4(t) - 2 \exp(ie_1p)\eta_p^2(t)$$
$$+ 1/2 \exp(ie_3p)\eta_p^6(t)] + \ldots]$$
$$\frac{\partial \eta_p^2(t)}{\partial t} = - M^{2j}_{-p,p}\eta_p^j(t) + \theta_p^2(t)$$
$$\frac{\partial \eta_p^3(t)}{\partial t} = -M^{3j}_{-p,p}\eta_p^j(t) - K[1/2 \exp(iU_{23}p)\eta_p^6(t) - 2 \exp(ie_2p)\eta_p^4(t)$$
$$+ 1/2 \exp(ie_1p)\eta_p^2(t)] + \ldots]$$
$$\frac{\partial \eta_p^4(t)}{\partial t} = -M^{4j}_{-p,p}\eta_p^j(t) + \theta_p^4(t) \tag{30}$$
$$\frac{\partial \eta_p^5(t)}{\partial t} = -M^{5j}_{-p,p}\eta_p^j(t) - K[1/2 \exp(iU_{31}p)\eta_p^5(t) - 2 \exp(ie_3p)\eta_p^6(t)$$
$$+ 1/2 \exp(ie_2p)\eta_p^4(t)] + \ldots]$$
$$\frac{\partial \eta_p^6(t)}{\partial t} = -M^{6j}_{-p,p}\eta_p^j(t) + \theta_p^6(t)$$

and for the transposed equations we have

$$
\begin{aligned}
\frac{\partial \eta^{t1}_{-p}(t)}{\partial t} &= -\eta^{tj}_{-p}(t) M^{j1}_{p,-p} + \theta^{t1}_{-p}(t) \\
\frac{\partial \eta^{t2}_{-p}(t)}{\partial t} &= -\eta^{tj}_{p}(t) M^{j2}_{p,-p} - K[1/2 \exp(iU_{31} p) \eta^{t5}_{-p}(t) - 2 \exp(ie_1 p) \eta^{t1}_{-p}(t) \\
&\quad + 1/2 \exp(ie_1 p) \eta^{t} \ldots] \\
\frac{\partial \eta^{t3}_{-p}(t)}{\partial t} &= -\eta^{tj}_{-p}(t) M^{j3}_{p,-p} + \theta^{t3}_{p}(t) \\
\frac{\partial \eta^{t4}_{-p}(t)}{\partial t} &= -\eta^{tj}_{p}(t) M^{j4}_{p,-p} - K[1/2 \exp(iU_{12} p) \eta^{t1}_{-p}(t) - 2 \exp(ie_2 p) \eta^{t3}_{-p}(t) \\
&\quad + 1/2 \exp(ie_2 p) \ldots] \\
\frac{\partial \eta^{t5}_{-p}(t)}{\partial t} &= -\eta^{tj}_{-p}(t) M^{j5}_{p,-p} + \theta^{t5}_{p}(t) \\
\frac{\partial \eta^{t6}_{-p}(t)}{\partial t} &= -\eta^{tj}_{p}(t) M^{j6}_{p,-p} - K[1/2 \exp(iU_{23} p) \eta^{t5}_{-p}(t) - 2 \exp(ie_3 p) \eta^{t5}_{-p}(t) \\
&\quad + 1/2 \exp(ie_3 p) \ldots]
\end{aligned}
\tag{31}
$$

After introduction of kernels as in the preceding section, we find

$$
\begin{aligned}
\frac{\partial \eta^{i}_{p}(t)}{\partial t} &= -N^{ij}_{-p,p} \eta^{j}_{p}(t) + \zeta^{i}_{p}(t) \\
\frac{\partial \eta^{ti}_{-p}(t)}{\partial t} &= -\eta^{tj}_{-p}(t) N^{ji}_{p,-p} + \zeta^{ti}_{p}(t)
\end{aligned}
\tag{32}
$$

where

$$
N = \begin{pmatrix}
0 & -A_1 & -a_3^*/2 & -C_2/2 & -a_2^*/2 & -B_3/2 \\
1 & 0 & 1/2 & -a_3/2 & 1/2 & -a_2/2 \\
a_3^*/2 & -B_1/2 & 0 & -A_2 & -a_1^*/2 & -C_1/2 \\
1/2 & a_3/2 & 1 & 0 & 1/2 & -a_1/2 \\
a_2^*/2 & -C_3/2 & a_1^*/2 & -B_2/2 & 0 & -A_3 \\
1/2 & a_2/2 & 1/2 & a_1/2 & 1 & 0
\end{pmatrix}
$$

and

$$
\begin{aligned}
A_i &= [1 + 2 \exp(ie_i p)] \\
B_i &= [1 - \exp(ie_i p)]
\end{aligned}
$$

$$C_i = [1 - \exp(i\varepsilon_{ijk}U_{jk}p)]$$

with no summation on the indices.

The correlation function of the noise is now

$$\langle \zeta_p^i(t)\zeta_{-p}^{tj}(t')\rangle_\zeta = 2\delta(t - t')(N^{-1})_{-p,p}^{ij}$$

In this kind of Hartree approximation the correlation functions can be calculated in the usual way:

$$\langle \eta_p^i \eta_{-p}^{tj}\rangle = \lim_{t \text{ and } t' \to \infty} \langle \eta_p^i(t)\eta_{-p}^{tj}(t')\rangle_\zeta = (N^{-1})_{-p'p}^{ij} \tag{33}$$

and we can finally write

$$\langle \eta_{\bar{r}}^i \eta_{\bar{r}'}^{tj}\rangle = 2 \int \frac{d^3p}{(2\pi)^3} \exp[ip(\bar{r} - \bar{r}')](N^{-1})_{-p,p}^{ij} \tag{34}$$

The expression for the matrix N^{-1} can be easily performed, but the result is too long and not of primary interest.

Finally, we insist that this kind of Hartree approximation is made not from a saddle point equation, but from an exact equation; this is the main point of this calculation.

5. CONCLUSION

The aim of this work is not to exhibit fundamentally new results, except for the kind of Hartree approximation, but only to show how stochastic quantization offers a frame for the study of statistical systems. We have also introduced some simplifications of the formalism. Under this form one can apply the variational approach developed in our preceding papers[8,9] to the study of more complicated systems such as the two-dimensional Ising model with an external magnetic field or three-dimensional Ising model. More generally, we now have a new exact stochastic equation for the Ising model at our disposal, a result which is not trivial particularly in three dimensions, and which can open new perspectives in the study of this type of model.

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REFERENCES

1. G. Parisi and Y. S. Wu, *Sci. Sinica* **24** (1981) 483.
2. F. A. Berezin, *The Method of Second Quantization*, Academic Press, New York (1966).

3. V. N. Popov, *Functional Integral in Quantum Theory and Statistical Physics*, Reidel, Dordrecht (1983).
4. S. Samuel, *J. Math. Phys.* **21** (1980) 2806.
5. C. Itzykson and J. M. Drouffe, *Théorie statistique des champs*, Inter Edition du CNRS (1989).
6. J. N. Negele and H. Orland, *Quantum Many Particles Systems*, Addison-Wesley, Reading, Massachusetts (1988).
7. P. H. Damgaard and H. Huffel, *Phys. Rep.* **152** (1987) 227.
8. Y. Grandati, A. Bérard, and P. Grangé, *Phys. Lett. B* **304** (1993) 298.
9. A. Bérard and Y. Grandati, *Int. J. Theor. Phys.* **38** (1999) 623–639.